Trigonometry

U.S. Army Corps of Engineers – Simulated Water Management Model (SWMM) Lesson Plan

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KEY WORDS

Amplitude
Function
Endogenous Variables
Exogenous Variables
Model
Period
Phase Shift
Standard Normal Distribution
System Dynamics Model
Uncertainty

1.0. Introduction

Some would argue that modeling is "where art and math collide". When artists start a new piece of work their goal is to capture, approximate, demonstrate or make a point about something they see going on in the world, and communicate that to an audience.

When a mathematician makes a model, s/he has the same goal. The goal is to create a function or set of functions that approximate, describe or help us better understand a process, system or something going on in the world.

Functions are the most fundamental tools in any modeler's or mathematician's toolbox. Despite being one of the first mathematical concepts we are introduced to, functions are often poorly understood by otherwise advanced students. In fact, a review of what functions are and are not is often the first topic introduced to students in college-level math courses.

One way to think about functions is as a machine, such as a meat grinder. One puts whole cuts of meat (the function's x's) into the grinder (the f) and retrieves the ground meat (the $f(x)$ or y).

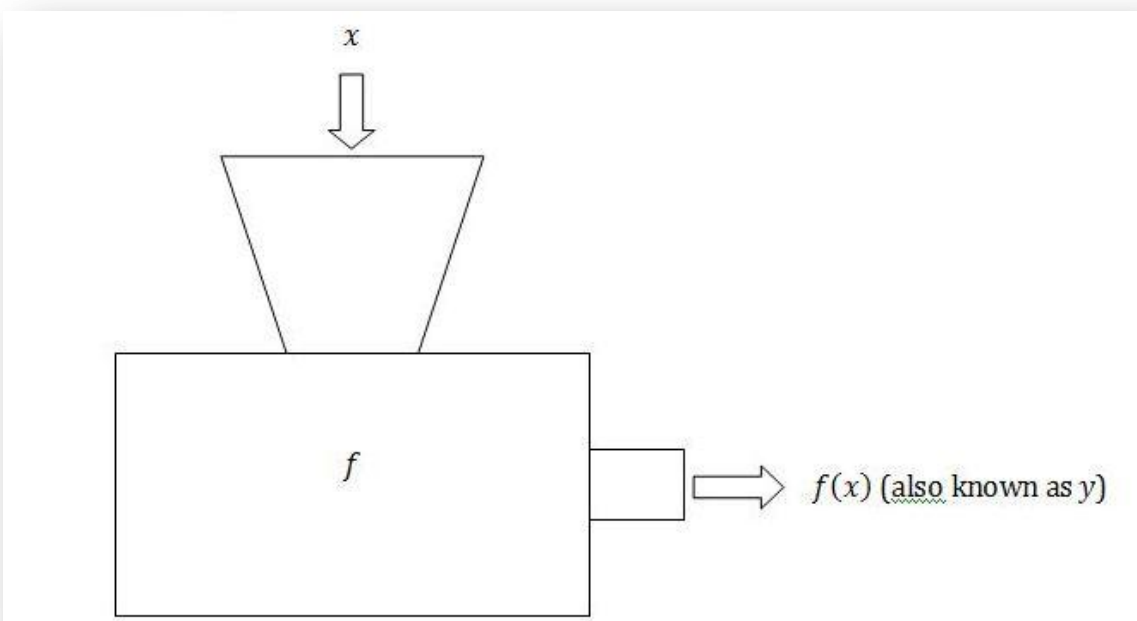

Figure 1 - Function Analogy

Perhaps the most important quality is that each x can produce only *one* y.

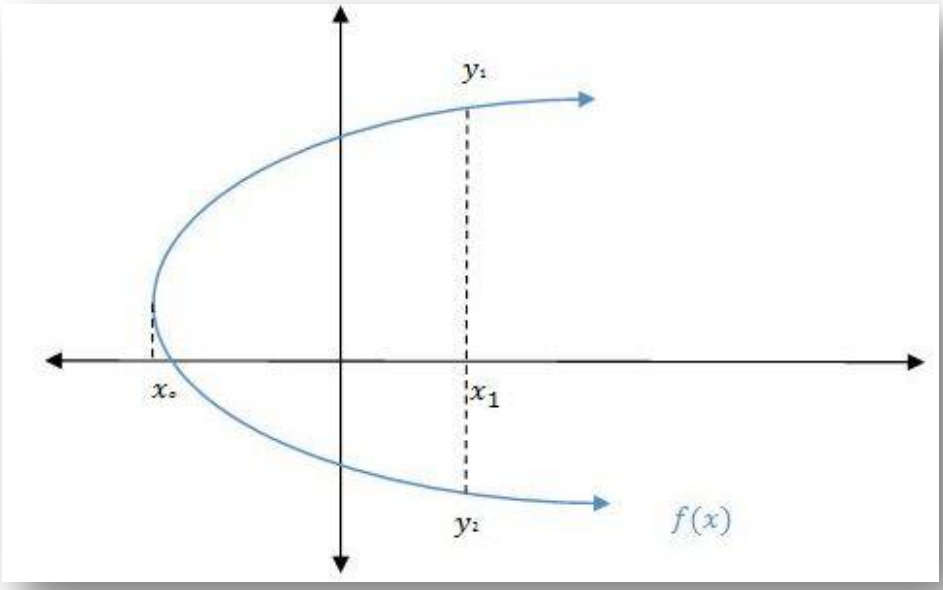

Figure 2 – Example Function #1

<u>Question</u>: Is this figure a function, yes or no?

<u>Answer</u>: No, x_1 produces both y_1 and y_2. For this reason $f(x)$ is only a function at a <u>single point</u>, namely x_o. Note, if we restrict our function's inputs (e.g. x's) to the interval from x_o to x_o (e.g. the single point) then $f(x)$ is a function.

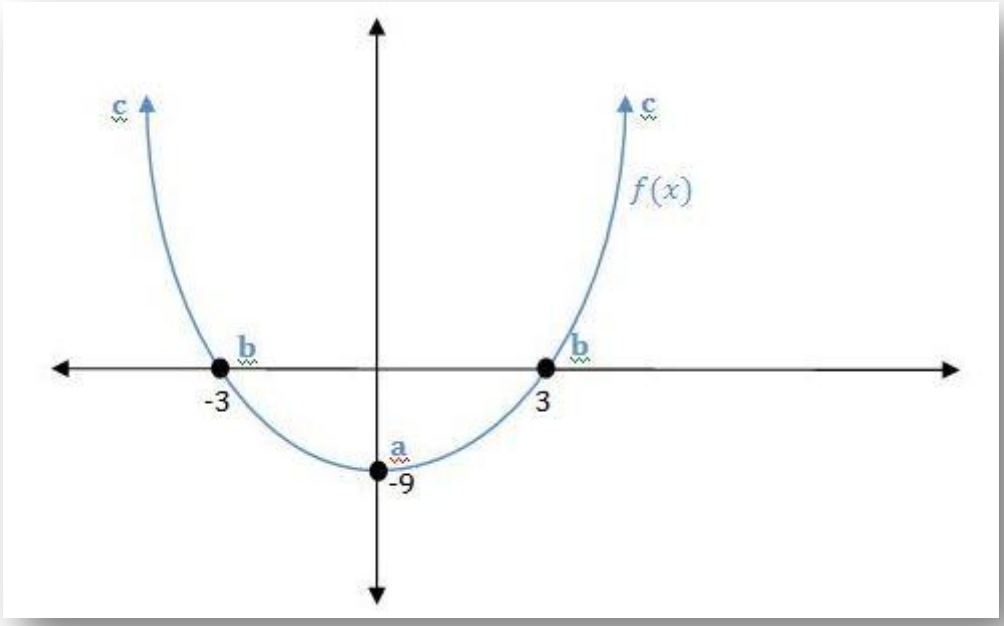

Figure 3 - Example Function #2

<u>Question</u>: Is this figure a function, yes or no?

<u>Answer</u>: Yes. In fact, $f(x) = (x+3)(x-3)$.

How do we know that? At **a**, where $x = 0, y = -9$. At points **b**, where $x = -3$ or $+3$, $y = 0$ (remember two x's producing the same y is O.K!). Finally, at points **c**, where $x \to +\infty, -\infty$ (e.g. as x gets really big or really small) $y \to +\infty, -\infty$ (e.g. also gets really big or really small). Or in words, as x approaches negative infinity, y approaches negative infinity and as x approaches positive infinity, y approaches positive infinity.

Finally, how does one know there would be two terms (e.g. no more and no less terms) in the polynomial equation (e.g. $(x+3)(x-3)$)? The graph turns over once. This suggests that the polynomial is "of degree" two, meaning its highest exponent is a squared term (e.g. x^2 , as in $f(x) = (x+3)(x-3) = x^2 - 9$).

1.1. Link to Curriculum

This module provides a primer through which students can begin to build a base of knowledge and intuition regarding trigonometry functions. The topics covered in this lesson are closely aligned with the Next Generation Science Standards, following elements of the National Resource Council (NRC) document, "A framework for K-12 Science Education": *MP.2, MP.3, and MP.4*

1.2. Summary of Model

In the provided model, students explore the science of modeling. The model they use is based on historical, observed precipitation over time. The goal is to teach students how to use a mathematical function (e.g. a sine function) to create a model that fits the historical data. In this case, the students will use a trigonomic function to create a model that accurately predicts rainfall for the Upper American River Watershed. This model is the fourth in a series of four water resource models created for middle and high school science, technology, engineering and mathematics (STEM courses). The final product of the trigonometry model (e.g. a model of rainfall for the Upper American River Watershed) is an *exogenous* input in the much larger Earth Sciences system dynamics lesson, model and game. As such, the lesson surrounding the trigonometry model can be expanded by introducing (or re-introducing) students to its use in the earth sciences model and game. Furthermore, the process that is used to produce the precipitation model (in the trigonometry and earth sciences models) is identical to the process used to produce the temperature input in the earth sciences model. Accordingly, the trigonometry lesson can be expanded to allow students the opportunity to apply the concepts in the precipitation model to another context. This is discussed in more detail in Section 2 below.

2.0. The Model

The final state of the trigonometry model is an accurate monthly prediction of rainfall (based on historical data) in the Upper American River Watershed. These predicted monthly rainfall totals are in turn used as *exogenous* inputs into a larger Earth Sciences system dynamics model, which explores the operations of flood risk management reservoirs. *Flood risk management reservoirs* are designed to store (in a lake) peak river flows which may otherwise result in flooding. By doing so river flow is temporary reduced and the stored water can be release more slowly and safely when the river flow is naturally lower. The fundamental question facing the water manager at a reservoir during a storm (or snowmelt) event is: how much water should be stored (or conversely, how much water should be released)? In order to answer this question water managers must make predictions about the inflows (river flows into the reservoir) based on the precipitation (or snowmelt) that they expect will occur. Thus, this model introduces students to the first (and perhaps most important) question facing water resource managers deciding how to build or operate a reservoir, "*Will it rain; and if it does rain, how much will it rain?*"

2.1. System Dynamics Models

The type of model used in this lesson is an example of a *system dynamics models* model built using software called Vensim. These models are growing in popularity and are used in the fields of economics, business management, life sciences, physical sciences and engineering, where they are used to explain complex and interconnected operations, organizations and/or processes; also known as *systems*. The system dynamics approach to understanding complex operations, organizations and processes (e.g. systems) involves modeling the system as an array of interrelated (i.e. endogenous) variables. System dynamics models use feedback loops and *stock and flow relationship*s to predict complex and otherwise confusing behaviors and outcomes.

In the model shown, the watershed over a four year period is the modeled system. Each word in the figure below represents a model variable. The arrows between the variables represent internal relationships between their values. The values of variables with arrows pointing toward them are dependent on the values of the variables to which they are connected. In other words, the values of these variables are determined *within* the system model. Any variable with an internally determined value is known as an *endogenous variable.* These are the variables with values predicted by the model. *Exogenous variables,* on the other hand, have values that are input into (instead of determined by) the model. The precipitation variable (produced by the trigonometry model and used in the displayed earth sciences model) is an example of an exogenous variable. The goal of any model is to predict the value of *dependent* or *endogenous* variables. Often the first and sometimes most important step in realistically modeling the behavior of endogenous variables is "calibrating" the models exogenous variables to match the observed data (or reality).

One of the strengths of the system dynamics approach is the ability of system dynamics models to replicate *stock and flow relationships*. In systems, models stocks are accumulations of flows. The water in a reservoir is an example. Water flows into a reservoir from a river and out of the reservoir through a dam. The difference between this inflow and outflow is accumulated or stored in the reservoir, which is a stock variable.

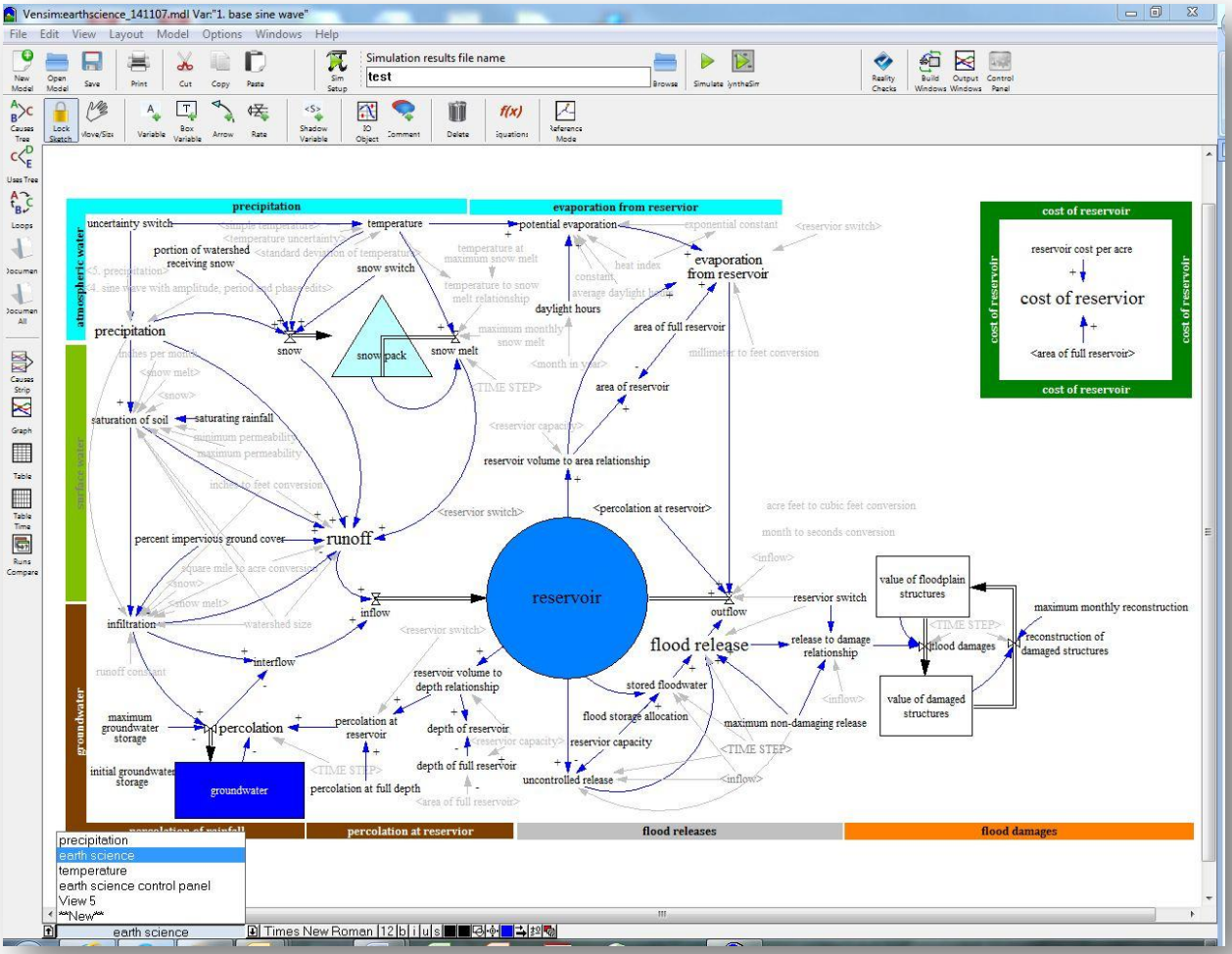

Figure 4 - System Dynamics Model (Earth Science Example)

System modeling is now its own field of study at a growing set of universities such as the Massachusetts Institute of Technology (MIT), California State University Chico (CSUC) and others.[1] The goal of the trigonometry model is not an introduction into *system dynamics modeling*. However, should students and or instructors be interested in learning more about the study, field or careers in system dynamics, a few useful links are provided in the footnote below.[2]

2.2. Model View

In the trigonometry model, each box or word corresponds with a variable. These variables' linkages with each other are mathematically defined through a system of equations (that can be viewed

[1] MIT program in system design and management: http://sdm.mit.edu/; CSU-Chico program in Business Information Systems: http://www.csuchico.edu/cob/prospective/explore/majors.shtml

[2] System Dynamics Society: http://www.systemdynamics.org/what-is-s/, Wikipedia page on system dynamics: http://en.wikipedia.org/wiki/System_dynamics.

using the process described in the user's manual) and symbolized by the arrows displayed on the model screen.

As is mentioned above, the model generates a prediction of monthly rainfall in the Upper American River Watershed, based on historical data. The table below displays the historical rainfall totals at a representative location within the watershed.

	Rainfall in Inches		
Month	Average	Max	Min
January	6.4	18.42	0.52
February	6.01	17.61	0.26
March	5.25	16.77	0.93
April	2.79	6.93	0.07
May	1.21	5.58	0.00
June	0.37	1.57	0.00
July	0.05	2.97	0.00
August	0.08	1.59	0.00
September	0.46	3.68	0.00
October	1.83	6.42	0.00
November	4.16	13.45	0.12
December	5.86	16.78	0.00
total	*34.47*		

Figure 5 - Historic Rainfall Data

The historic data is also shown graphically in model and below (in Figure 6).

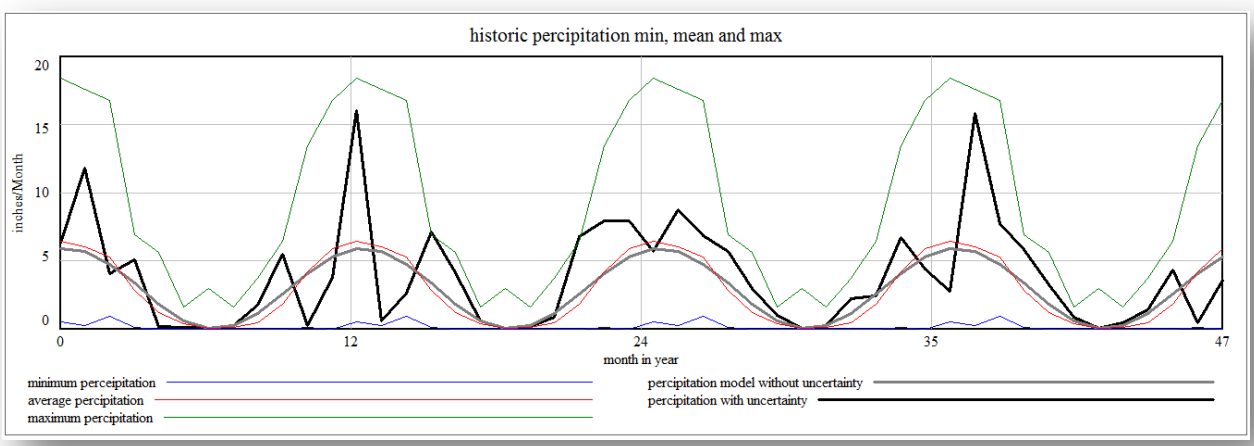

Figure 6 - Historic Rainfall Graph

2.2.1. Dashboard SyntheSim Mode

By pressing the SyntheSim button, the constant variables can be toggled to dynamically display important behaviors produced by the variables in the model. Variables of interest are color coded by topic (e.g. amplitude, phase shift, uncertainty adjustments, etc…) and can be adjusted within a simulation of the model (the model can be simulated using the procedure described in the user's manual). Their dynamic impact on outcome variables can be viewed in the provided graphs. Students should focus on gaining an intuitive and dynamic understanding of how each variable of interest impacts the model. In other words, they should be able to answer the following questions in association with the variables covered under each topic within the model (e.g. amplitude, phase shift, etc…)

> *Example Question: What is the impact of a positive/negative change in the variable on $f(x)$ (e.g. y)?*
>
> *Example Question: Why does the selected value (for each variable) produce a more accurate prediction of rainfall ((e.g. why does the value of the variable produce more accurate y values)?*

2.2.2. Equation One

The graph and variable "1. base sine wave" in the model (also displayed in Figure 7 below) show a basic sine wave. This base function is edited in the model by adjusting its: a) amplitude (height) and vertical shift (maximum and minimum values in the "2. sine wave with amplitude edit" variable); b) *period* (time it takes to complete a cycle, in the "3. sine wave with amplitude and period edits" variable); and c) phase (position at the first time period, in the "4. sine wave with amplitude, period and phase edits" variable).

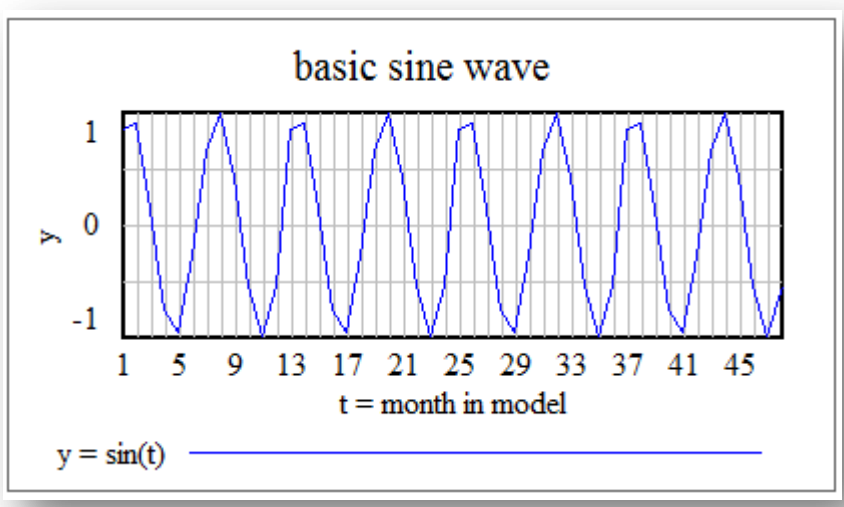

Figure 7 - Base Sine Wave

The goal of the adjustments outlined above is to produce a sine wave that accurately models *average* monthly rainfall in the Upper American Watershed. Further adjustments are made to create a plausible model of monthly rainfall; since actual monthly rainfall rarely matches the average). Given this goal, it is worth noting some of the strengths and weakness of the sine function, in association with our desired goal. The sine function oscillates, returning to the same value every 2π time periods. While this period of oscillation (e.g. $2\pi \cong 6.24$) is incorrect, average monthly rainfall does oscillate in a similar (e.g. wave-like fashion), returning to the same value every 12 months. The range of values the function takes (e.g. from -1 to +1), which in our model should reflect monthly rainfall totals, needs adjustment. The phase of the function (e.g. the value at t=0) is nearly, but not totally correct.

2.2.3. Equation Two

The sine function produces a base "y" value (e.g. the value of the sine function without any edits, in the equation $y = sin(t)$). We make two edits to the base sine function in the "2. sine function with amplitude edit" variable.

1. We multiply this number by an "amplitude" factor that increases (or decreases) the base "y" value. For instance, if we multiply $f(x)$ by 2 (e.g. choose an amplitude of 2), it will double every base value the sine function produces (except 0). In this case, we choose amplitude based on the maximum and minimum rainfall (e.g. we want the height of the sine function to equal the difference between the wettest and driest months).

   Note, in this lesson and model we use the term *amplitude* to mean half the distance from the function's maximum to minimum value. This corresponds with the most common usage of the term as it relates to trigonomic functions. However, it is worth noting that the most accurate definition of amplitude is simply the change in a periodic function's (such as a sine function) value over a measure of time.

2. Next, we add the "amplitude" to the end of the function. Without this addition, half the y values will be negative and half will be positive (since the base sine function oscillates between a value of -1 and +1). By adding the amplitude, we ensure the minimum y value is the amount of precipitation that falls in the driest month, and the maximum y value is the amount of rain that falls in the wettest month.

The graph below (Figure 8), also shown in the model, shows the difference between the base sine function and our edited function, which includes amplitude adjustments. **Students can adjust the wettest and driest month slider variables to view the impact of these variables on the function.**

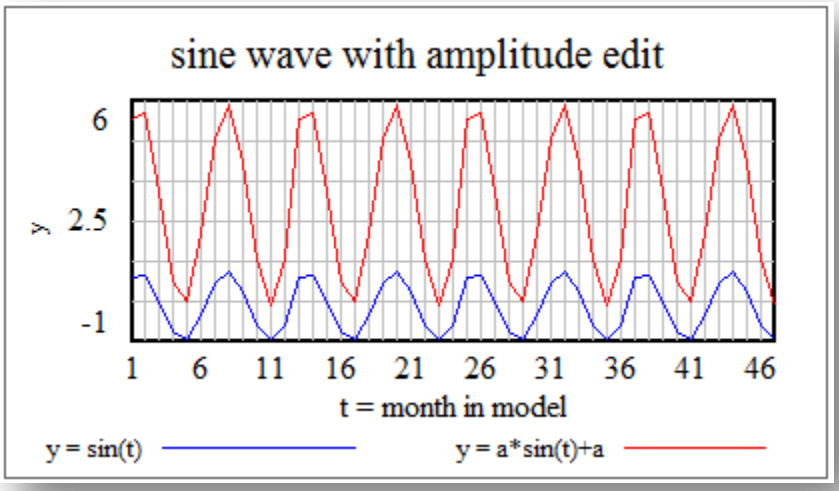

Figure 8 - Sine Wave with Amplitude Edits

2.2.4. Equation Three

The basic sine wave, $y = sin(t)$, takes $t = 2\pi \cong 6.28$ time periods to complete each cycle (e.g. the period of time it takes along the x-axis to oscillate between its maximum and minimum values. In order to adjust this function, $f(x) = \sin(t)$ must be multiplied by some value, (e.g. $f(x) = \sin(p \times t)$), to slow the speed of the function down to about half speed. More precisely, it takes 6.28 time periods for $f(x) = \sin(t)$ to complete a cycle; but we want it to complete a cycle in 12 months, or equivalently 12 time periods). Thus, the multiplier used, $p = \frac{2\pi}{12}$, slows the function down so that it takes precisely 12 time periods (months) to complete a precipitation cycle. Our updated function, $f(x) = a * \sin\left(\frac{2\pi}{12}t\right) + a$ (where a denotes the amplitude adjustments made in equation 2 described above), takes precisely one year to complete a precipitation cycle. Note that all that has really been done with this multiplier is to reduce the value of t in $y = sin(t)$, which naturally slows the function down (since its speed is determined by the value of t).

Figure 9 (also shown in the model) displayed below shows the value of our base sine function, the sine function with the amplitude edits described in section 2.2.3; and the sine function with amplitude and period adjustments.

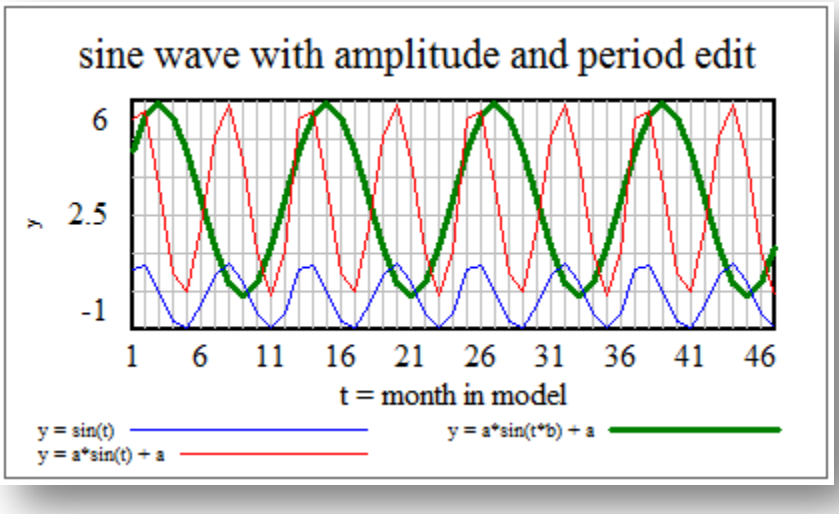

Figure 9 - Sine Wave with Amplitude and Period Edits

2.2.5. Equation Four

The basic sine function hits its highest point where $t = \frac{\pi}{2}$ and once every 2π thereafter. To accurately model rainfall, the function must be shifted horizontally so that this occurs in late January (e.g. at month in year equals 1.75 or time equals 0.75; since the model begins when time equals 0). This amount will be added to the value of t in the base sine functions, $y = sin(t)$. Note that in equation three a period multiplier was added to "slow t down". Therefore, 1.75 times the same multiplier must be multiplied in order to ensure our precipitation function hits its maximum value roughly every January 20th.

Thus, our updated equation is $y = a \times \sin\big(p(t + 0.75)\big) + a$, where $p = \frac{2\pi}{12}$ the period from equation three and a is the amplitude determined in equation two.

2.2.6. Equation Five

Equation four produces a model of average monthly rainfall. Statistically speaking, this is the most likely rainfall for any given month. However, the actual rainfall in any given month often deviates (e.g. is higher or lower) from the average. In other words, the precipitation (e.g. rainfall) that falls in any given month is unpredictable. A normal distribution is used to model the uncertainty in the actual rainfall data (see Figure 10 below). This is done by allowing the computer to randomly select a number from a *standard normal distribution* (shown below). The center of this distribution represents the average (also the median and mode). Any value selected (by the computer) to the left of the center generates a rainfall amount below the mean. Any value selected (by the computer) to the right of the center will generate a rainfall amount above the monthly average. They y-axis of the figure describes the probability (used by the computer) of selecting the associated x value. The values on the x-axis (of the figure below) display the number of *standard deviations* above or below the mean. In the case of this model, a *standard deviation* is approximately equal to the average monthly rainfall. More than two-thirds of the values selected by the computer will be within one

standard deviation of the mean generated by our model, meaning that more than two-thirds (68 percent) of the values selected by the computer will be between zero inches (e.g. one standard deviation below the mean) and double the average (e.g. one standard deviation below the mean). Ninety-five percent of the values selected by the computer are within two standard deviations of the mean and ninety-nine percent are within three standard deviation of the mean.

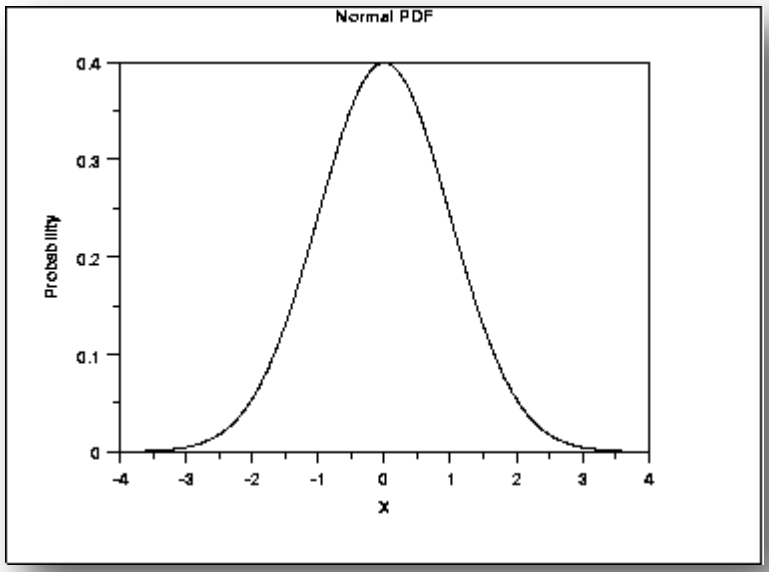

Figure 10 - Normal Distribution

In this model, any number (selected by the computer) below -1 or above +3 is corrected to be equal to -1 or +3. This prevents our model from generating negative amounts of rainfall (e.g. rainfall more than one standard deviation below the mean) or rainfall amounts that are more than triple that month's average.

The graph displayed below (Figure 3; also shown in the model) shows the model's predicted rainfall with uncertainty (black line), without uncertainty (grey line) as well as the historical average, minimum and maximum month totals (red, purple and green lines – respectively).

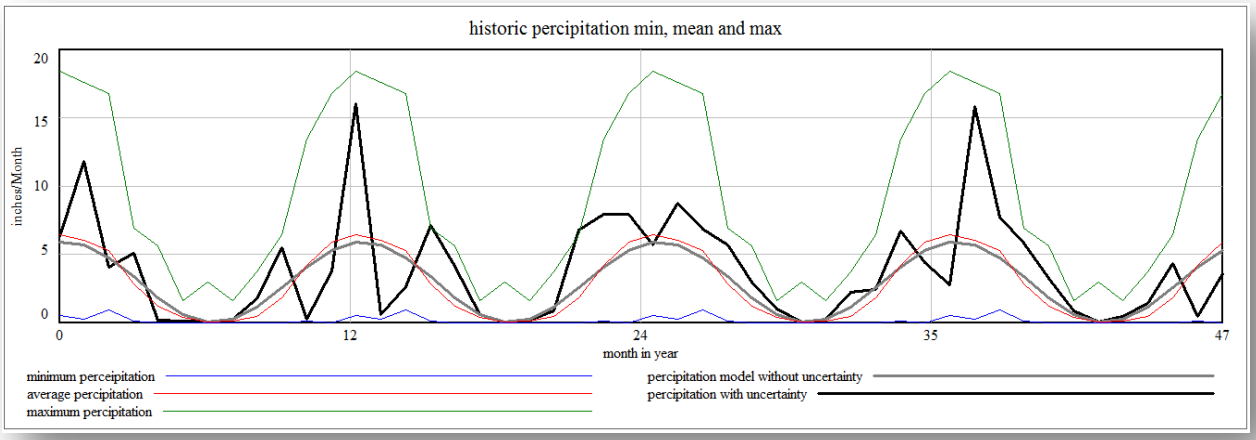

Figure 11 - The Model's Fit to Historical Data

3.0. Discussion Topics and Follow-on Activities

The following activities and discussion topics provide students and teachers with an opportunity to practice the skills learned in this model, view their application in the context of a larger system dynamics model and further expand upon the topics covered in the trigonometry model.

3.1. Temperature Model

As a follow-on to this model, students could attempt to produce a sine function that models the temperature data displayed in the table below. They could check this temperature model against the temperature model displayed in the "temperature view" of the earth sciences model.

	Temperature in Celsius		
Month	Average	Maximum	Minimum
January	7	20	-7
February	9	24	-2
March	11	29	-2
April	14	35	-1
May	17	40	2
June	22	42	7
July	25	45	11
August	25	40	11
September	22	39	4
October	18	36	3
November	12	27	-3
December	8	19	-5

Figure 12 - Temperature Table

3.2. Earth Sciences Model

As is discussed above, this precipitation model serves as an exogenous input into the earth sciences model. As a follow-on activity, the students could review the precipitation data used and explore its impact on the larger earth sciences model. The User's Guide and the earth sciences lesson could be used to lead this exploration.

3.3. Discussion Topics

Some sample questions and answers for use in discussion with the students or student write-ups are as follows:

> *Example Question: As is discussed above, the manipulations of the sine function (e.g. equations one through four) produce an estimate of average or mean monthly rainfall. While the average rainfall is statistically the most likely amount of rain in any given month, actual rainfall is often above or below this average. As a result, a normal distribution was used to model the randomness (e.g. uncertainty) in the actual rainfall data. When is the normal distribution not the right model for uncertainty in the actual monthly data?*
>
> *Example Answer: The normal distribution is symmetrical about the mean. As a result, the normal distribution will only produce a "good" representation of actual uncertainty when above average and below average amounts of rainfall are equally likely (see Figure 13 below). In California (and many other places), there are some months of the year where actual precipitation is more likely to be below or above the mean. For instance, in the summer months when average rainfall is near zero, the probability of above average rainfall exceeds the probability of below average rainfall (since rainfall totals cannot be negative). The normal distribution does not fit those months.*

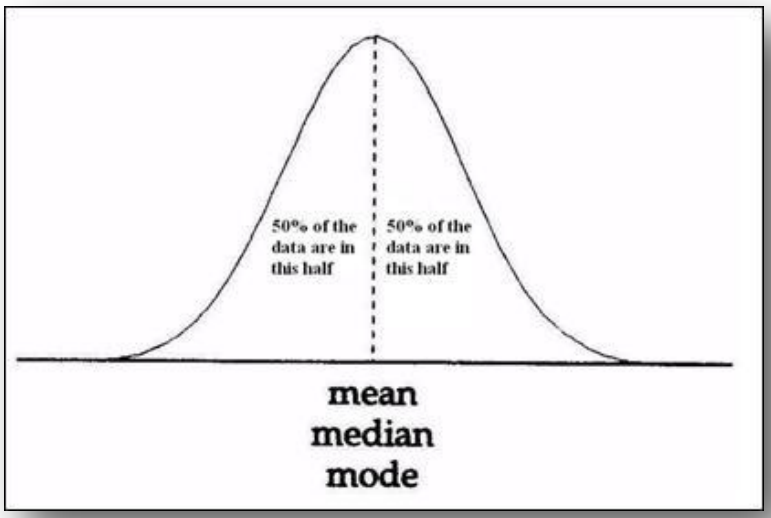

Figure 13 - Symmetry of Normal Distribution

Example Question: When does the average precipitation model produce its worst and best predictions?

Example Answer: The model will make a good prediction over the entire year, but will not necessarily make good predictions in shorter time periods where there is extreme weather such as drought or extreme precipitation events. In general, it will work best during typically wet and dry times. Within the year, the model produces its best predictions where the red and grey lines in Figure 11 match (or nearly match); it produces its worst predictions where they are far apart.

Example Question: When would you want to use this model/not use this model?

Example Answer: This model works relatively well for locations that do not often experience extremes in precipitation. It will also work well when a general prediction of the average rainfall is what is needed. However, often times a model of flood (rainfall far above the mean) or drought (many months in a row of below average rainfall) is what is needed. In these circumstances, the application of this model would be inappropriate.

Example Question: What would be a good quantitative way of measuring how well the average rainfall model fits the data?

Example Answer: For the average rainfall model, the model's predicted values could be subtracted from the actual average values (shown in Figure 5). Positive values would represent predicted values that are too low (e.g. below the actual mean), while negative values would represent predicted values that are too high (e.g. above the actual mean).

4.0 Conclusion

This trigonometry lesson and model gives students a real world example of trigonometry being used to predict monthly rainfall totals. The rainfall totals are in turn used as *exogenous* inputs into a larger Earth Sciences system dynamics model that explores the operations of flood risk management reservoirs. *Flood risk management reservoirs* are designed to store (in a lake) peak river flows that may otherwise result in flooding. By doing so, river flow is temporarily reduced, and the stored water can be released more slowly and safely when the river flow is naturally lower. The fundamental question facing the water manager at a reservoir during a storm (or snowmelt) event is: how much water should be stored (or conversely, how much water should be released)? In order to answer this question, water managers must make predictions about the inflows (river flows into the reservoir) based on the precipitation (or snowmelt) that they expect to occur. Thus, this model introduces students to the first (and perhaps most important) question facing water resource managers deciding how to build or operate a reservoir, "*Will it rain; and if it does rain, how much will it rain?*"